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Calculus I

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First Exam

October 3, 2014, 11:00-11:50am.

Problem	Possible points	Score
1	20	20
2	30	30
3	20	20
4	10	10
5	20	20
Total	100	100

Exercise 1. (20p) Find the following limits

(a)

$$\lim_{x\to 5} \frac{(x-5)\sin\left(\frac{\pi x}{20}\right)}{\sqrt{x+4}-3}.$$

$$\lim_{x\to 5} \frac{(x-5)\sin\left(\frac{\pi x}{20}\right)}{\sqrt{x+4}-3} =$$

$$\lim_{x\to 5} \frac{(x-5)\sin\left(\frac{\pi x}{20}\right)\left(\sqrt{x+4}+3\right)}{\left(\sqrt{x+4}-3\right)\left(\sqrt{x+4}+3\right)} =$$

$$\lim_{x\to 5} \frac{\cancel{(x-5)}\sin\left(\frac{\pi x}{20}\right)\left(\sqrt{x+4}+3\right)}{\cancel{x-5}} =$$

$$\lim_{x\to 5} \sin\left(\frac{\pi x}{20}\right)\left(\sqrt{x+4}+3\right) =$$

$$\sin\left(\frac{\pi\cdot 5}{20}\right)\left(\sqrt{5+4}+3\right) =$$

$$\sin\left(\frac{\pi}{4}\right)\cdot 6 = \frac{\sqrt{2}}{2}\cdot 6 = \boxed{3\sqrt{2}}$$

(b)

$$\lim_{x\to\infty} \frac{\sqrt{2x^4+1}+\sin(2x)}{x^2+25x+2014}.$$

$$\lim_{x\to\infty} \frac{\sqrt{2x^4+1} + \sin(2x)}{x^2 + 25x + 2014} =$$

$$\lim_{x\to\infty} \frac{\frac{\sqrt{2x^4+1}}{x^2} + \frac{\sin(2x)}{x^2}}{1 + \frac{25}{x} + \frac{2014}{x^4}} =$$

$$\lim_{x\to\infty} \frac{\sqrt{2+\overbrace{\frac{1}{x^4}}^{\to 0}} + \overbrace{\frac{\sin(2x)}{x^2}}^{\to 0}}{1 + \underbrace{\frac{25}{x}}_{\to 0} + \underbrace{\frac{2014}{x^4}}_{\to 0}} = \boxed{\sqrt{2}}$$

Exercise 2. (30p)

(a) For what value of a is the function continuous

$$f(x) = \begin{cases} \frac{6x^2}{\sin(x^2)} & \text{if } x > 0, \\ (a^2+1)\sin x + 3a & \text{if } x \leq 0. \end{cases}$$

f is continuous for $x \neq 0$ - obvious.

It is continuous at $x = 0$ iff

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\lim_{x \to 0^-} f(x) = (a^2+1)\sin 0 + 3a = 3a$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 6 \cdot \frac{1}{\frac{\sin(x^2)}{x^2}} = 6$$

$$f(0) = 3a$$

$$3a = 6$$

$$\boxed{a = 2}$$

The function f is continuous if and only if $a = 2$.

(b) Use the definition to find the derivative of $f(x) = \sqrt{3x+5}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$\lim_{h \to 0} \frac{\sqrt{3(x+h)+5} - \sqrt{3x+5}}{h} =$$

$$\lim_{h \to 0} \frac{\left(\sqrt{3(x+h)+5} - \sqrt{3x+5}\right)\left(\sqrt{3(x+h)+5} + \sqrt{3x+5}\right)}{h\left(\sqrt{3(x+h)+5} + \sqrt{3x+5}\right)} =$$

$$\lim_{h \to 0} \frac{(3(x+h)+5) - (3x+5)}{h\left(\sqrt{3(x+h)+5} + \sqrt{3x+5}\right)} =$$

$$\lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3(x+h)+5} + \sqrt{3x+5}\right)} =$$

$$\frac{3}{\sqrt{3(x+0)+5} + \sqrt{3x+5}} = \boxed{\frac{3}{2\sqrt{3x+5}}}$$

(c) Find vertical and horizontal asymptotes of $f(x) = \dfrac{(x-2)\sin(x^2+1)}{x^3-5x^2+6x}$.

$$\frac{(x-2)\sin(x^2+1)}{x^3-5x^2+6x} = \frac{\cancel{(x-2)}\sin(x^2+1)}{x\cancel{(x-2)}(x-3)} =$$

$$\frac{\sin(x^2+1)}{x(x-3)}$$

The denominator equals 0 at $x=0$ and $x=3$, but $\sin(0^2+1) \neq 0$, $\sin(3^2+1) \neq 0$ so the lines

$\boxed{x=0 \text{ and } x=3 \text{ are vertical asymptotes}}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\overbrace{\sin(x^2+1)}^{\to \text{ bounded}}}{\underbrace{x(x-3)}_{\to\infty}} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\overbrace{\sin(x^2+1)}^{\text{- bounded}}}{\underbrace{x(x-3)}_{\to\infty}} = 0$$

so

$\boxed{y=0 \text{ is the horizontal asymptote}}$

Exercise 3.(20p) Find the derivative of
(a)

$$\cos\left(\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}\right).$$

$$\cos\left(\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}\right)' =$$

$$-\sin\left(\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}\right)\left(\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}\right)' =$$

$$-\sin\left(\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}\right)\frac{1}{2\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}}\left(x+\sin\left(\frac{1}{\sqrt{x+1}}\right)\right)' =$$

$$-\sin\left(\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}\right)\frac{1}{2\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}}\left(1+\cos\left(\frac{1}{\sqrt{x+1}}\right)\underbrace{\left(\frac{1}{\sqrt{x+1}}\right)'}_{\left((x+1)^{-1/2}\right)'}\right) =$$

$$\boxed{-\sin\left(\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}\right)\frac{1}{2\sqrt{x+\sin\left(\frac{1}{\sqrt{x+1}}\right)}}\left(1+\cos\left(\frac{1}{\sqrt{x+1}}\right)\left(-\frac{1}{2}(x+1)^{-3/2}\right)\right)}$$

(b)

$$\frac{\cos^2 x - x}{2 + \tan x}.$$

$$\left(\frac{\cos^2 x - x}{2 + \tan x}\right)' =$$

$$\frac{(2\cos x(-\sin x)-1)(2+\tan x) - (\cos^2 x - x)\frac{1}{\cos^2 x}}{(2+\tan x)^2}$$

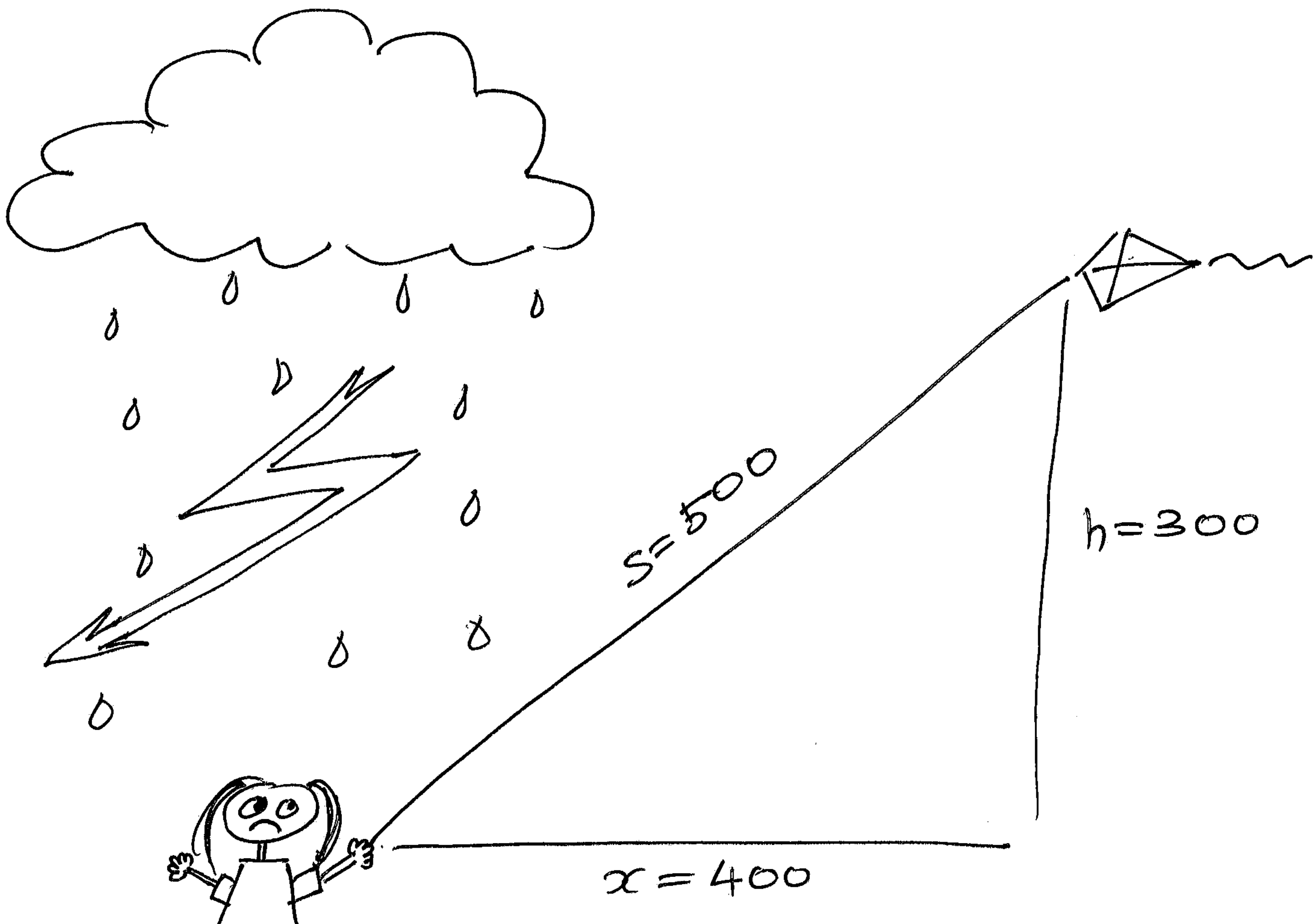
Exercise 4.(10p) Find dy/dx if $y^2 = x^2 + \sin(xy)$.

$$2y\frac{dy}{dx} = 2x + \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

$$\left(2y - x\cos(xy)\right)\frac{dy}{dx} = 2x + y\cos(xy)$$

$$\boxed{\frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}}$$

Exercise 5. (20p) A girl files a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?



$$\sqrt{500^2-300^2} = \sqrt{(5^2-3^2)\cdot 100^2} = 4\cdot 100 = 400$$

$$s^2 = x^2 + h^2$$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + \underbrace{\frac{dh^2}{dt}}_{}$$

0 because $h=300$ does not change in time.

$$\frac{ds}{dt} = \frac{x\frac{dx}{dt}}{s} = \frac{400\cdot 25}{500} = \frac{4\cdot 25}{5} = 4\cdot 5 = 20\ \mathrm{ft/sec}$$